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M/M/1 MULTICLASS FIFO QUEUES AND GENERALIZATIONS.

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Abstract. The multiclass open queue with Poisson arrivals, exponential services and FIFO scheduling is considered. Arrival and service rates depend on the class of the customer. We derive the semi-explicit expressions of the most important steady state distributions of this model. We prove a theorem about the insensitivity of these distributions with respect to a set of class-independent scheduling disciplines. Numerical aspects are also discussed. As a secondary result, the semi-explicit expression of the distributions associated to the single class M/H/1 queue are exhibited.

FILES D'ATTENTE M/M/1 FIFO MULTICLASSE ET GENERALISATIONS.

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Résumé. On considère la file d'attente ouverte avec arrivées poissonniennes, services exponentiels et discipline PAPS. Les taux d'arrivée et de service dépendent de la classe du client. On obtient les expressions semi-explicites des principales distributions stationnaires du modèle. On prouve un théorème d'insensibilité de ces distributions par rapport à un ensemble de disciplines indépendantes des classes des clients. Les aspects numériques sont aussi discutés. Comme conséquence de ce travail, on obtient les expressions semi-explicites des distributions stationnaires pour la file monoclasse M/H/1.

1. Introduction.

We consider a multiclass single server queue the scheduling of which is FIFO without any class distinction. There are K classes denoted 1,2,...,K and we note [1,K] the set {1,2,...,K}. For any k in [1,K], the arrival stream of the class-k customers is Poisson and the K processes are independant. Customer's service times depend on their classes ; they are exponentially distributed, independant of each other and of the arrival processes. The capacity of the queue is infinite, that is, we deal with an open system. This model will be denoted $M^K/M^K/1$ and we will be concerned only with its steady state behaviour. In the sixties, it was studied by Ancker and Gafarian [1] and their work was pursuived by Basharin [2]. The main result was the exhibition of numerical algorithms to compute several state probabilities. Recently, Botcharov extended these studies to general service times [3].

The main result of this paper is the semi-explicit form of the most important steady state distributions associated with the system. The obtained expressions are simple and easy to use numerically. The starting point is however different than those of [1],[2] or [3]. We will begin with the relation between the studied system and its natural associated single class hyperexponential queue. It is intuitively clear that if the classes of the customers are not taken into account, the $M^K/M^K/1$ queue "behaves" as a M/H/1 single class model, although this fact does not seem to have been exploited yet to derive informations for the multiclass context. A list of some of the principal notations is given in the Annex. The paper is organized as follows. In the next Section we state the relations between the steady state distibutions of the two related queues. In Section 3 a detailed study of the M/H/1 queue is done ; it is applied to the multiclass model in Section 4. In Section 5 the numerical utilization of the obtained expressions is discussed. Section 6 is for completeness : the mean values of the principal random variables associated with the multiclass queue are derived. Section 7 deals with the "insensitivity" of the steady state distributions with respect to the discipline in the queue (in a set of class independant scheduling disciplines). Finally, in Section 8 the straightforward extension to multiclass systems with hyperexponential service times (that is $M^K/H^K/1$ models) is exhibited.

2. The $M^K/M^K/1$ system and its related M/H/1 queue.

Let us note :

λ_k : arrival rate of class k customers,

$\mu_k = \dfrac{1}{S_k}$: service rate of class k customers.

$$\lambda = \sum_{k=1}^{K} \lambda_k \text{, the total arrival rate,}$$

$$\alpha_k = \frac{\lambda_k}{\lambda} \text{, the probability for a given arrival to be in class k.}$$

The natural single class model associated with the $M^K/M^K/1$ one is the M/H/1 queue with arrival rate λ and pdf (probability density function) of the service time given by

$$\sum_{k=1}^{K} \alpha_k \mu_k \exp(-\mu_k t)\,, \qquad \text{for } t > 0$$

Let us introduce the following set of states corresponding to the $M^K/M^K/1$ queue, states which will be called "micro-states" :

$$S = \{ (o) \} \cup \{ (c_1, c_2, \ldots, c_n) / \; n \geq 1 \text{ and } c_i \in [1,K] \text{ for } i=1,2,\ldots,n \}$$

where c_i is the class of the i^{th} customer in the (class independent) FIFO order (c_1 is the class of the customer being served) and (o) represents the empty system.

Over S the evolution of the system is an irreducible Markov process. In correspondance with each micro-state, let us consider the "regular" state $(n_1, n_2, \ldots, n_K)$ where n_k is the number of class-k customers in the system. That is, consider the operator T defined by :

$$T.(o) = \bar{o} = (o, o, \ldots, o)$$

$$T.(c_1, c_2, \ldots, c_n) = \bar{n} \quad \text{with } \sum_{i=1}^{n} ¶_{\{c_i = k\}} \,,$$

where $¶_X$ denotes the characteristic function of the set X.

For the single class M/H/1 queue defined above, we will consider the usual irreducible Markov process over states (n,k), $n \geq 1$ and k in [1,K]. The process is in the state (n,k) when n customers are in the system <u>and</u> the server is in phase k ; the empty system will be again represented by (o).

We will note p(.) a steady state probability of the multiclass model and q(.) a steady state probability of the single class one. We can now set the following theorem which establishes the relation between the stationary state distributions of the two systems.

Theorem 2.1

The two systems $M^K/M^K/1$ and M/H/1 defined above are ergodic under the same following necessary and sufficient condition :

$$\sum_{k=1}^{K} \frac{\lambda_k}{\mu_k} < 1$$

In this case we have :

$$p(o) = q(o)$$

$$(2.1) \qquad p(k, c_2, \ldots, c_n) = \overline{\alpha}^{\,\overline{n} - \overline{e_k}}.q(n,k) \qquad \text{with}$$

$$\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K), \qquad \overline{n} = T.(k, c_2, \ldots, c_n),$$

$$\text{for } \overline{u}, \overline{v} \in \mathbf{R}^K \ \overline{u} > 0 : \ \overline{u}^{\overline{v}} = u_1^{v_1} u_2^{v_2} \ldots u_K^{v_K},$$

and $\overline{e_k}$ the k^{th} vector of the canonical base of $\mathbf{R}^K$, that is $\overline{e_k} = (\delta_{1\kappa}, \delta_{2\kappa}, \ldots, \delta_{K\kappa})$.

Proof :

If E_1 and E_2 are respectively the set of equilibrium equations for the $M^K/M^K/1$ and the M/H/1 queues, we only have to check that, if q(.) is a normalized solution of the set E_2, then the function p(.) defined by (2.1) is a normalized solution of E_1. Since E_2 is the set of equilibrium equations of a M/H/1 system, it has a single positive normalized solution q(.) if and only if the well known condition $\lambda.S < 1$ holds, where S is the expectation of the hyper-exponential service time. We have :

$$\lambda.S = \lambda.\sum_{k=1}^{K} \frac{\alpha_k}{\mu_k} = \sum_{k=1}^{K} \frac{\lambda_k}{\mu_k}$$

So, the solution of E_1 is the given function p(.) and it is unique as E_1 corresponds to an irreducible Markov process.

The set E_1 is :

$$(2.2) \qquad p(o).\lambda = \sum_{j=1}^{K} p(j).\mu_j$$

$$(2.3) \qquad p(\overline{x}).(\lambda + \mu_k) = p(\overline{z}).\lambda_{c_n} + \sum_{j=1}^{K} p(j,\overline{x}).\mu_j$$

where $\overline{x} = (k, c_2, \ldots, c_n)$, $n \geq 1$,

and $\overline{z} = (k, c_2, \ldots, c_{n-1})$ if $n \geq 2$, $\overline{z} = (o)$ if $\overline{x} = (k)$.

The equations E_2 are :

(2.4) $$q(o).\lambda = \sum_{j=1}^{K} q(1,j).\mu_j$$

(2.5) $$q(n,k).(\lambda + \mu_k) = q(n-1,k).\lambda + \alpha_k \sum_{j=1}^{K} q(n+1,j).\mu_j$$

where $n \geq 1$ and, by convention, $q(o,k) = \alpha_k q(o)$.

By replacing p(.) in E_1 by the expression given in (2.1) we can easily check after some algebra that E_1 is satisfied if q(.) is a solution of E_2 and that p(.) is normalized if q(.) is normalized.

End of proof 2.1.

We can remark that in (2.1) the expression of $p(c_1,\ldots,c_n)$ does not depend on the order of the classes of the customers in positions 2,3,...,n. We have then by summation the following corollaries :

Corollary 2.2 :

Over regular states, the steady state distribution for the $M^K/M^K/1$ system is given by :

(2.6) $$p(\overline{n}) = C(\overline{n}).\overline{\alpha}^{\overline{n}}.\frac{1}{n}.\sum_{k=1}^{K} \frac{n_k}{\alpha_k}.q(n,k) \qquad \text{for } \overline{n} \neq \overline{o}$$

(2.7) $$p(\overline{o}) = q(o)$$

where $C(\overline{m})$ is the multinomial coeficient corresponding to vector $\overline{m}$, that is :

$$C(\overline{m}) = \begin{pmatrix} m \\ m_1\ m_2 \ldots m_K \end{pmatrix} = \frac{m!}{m_1!.m_2!.\ldots m_K!}$$

Proof :

We compute first the following probability :

$p(\overline{n};k)$ = steady state probability that the system is in regular state $\overline{n} \neq \overline{o}$
and that the customer being served is of class k

$$p(\overline{n};k) = \sum_{\substack{c_2,c_3,\ldots,c_n / \\ T.(k,c_2,\ldots,c_n) = \overline{n}}} \overline{\alpha}^{\,\overline{n} - \overline{e_k}} q(n,k)$$

$$= \overline{\alpha}^{\,\overline{n} - \overline{e_k}} q(n,k).C(\overline{n} - \overline{e_k})$$

Thus, we have :

$$(2.8) \qquad p(\overline{n};k) = \overline{\alpha}^{\,\overline{n}}.C(\overline{n}).\frac{n_k}{n.\alpha_k}.q(n,k)$$

and by summation over k equation (2.6) is obtained.

<u>End of proof 2.2.</u>

Corollary 2.3

If the state considered for the $M^K/M^K/1$ system is (n), i.e. the total number of customers, we can formally establish the intuitive identity

$$p(n) = q(n) \quad \text{for all } n \in \mathbf{N}.$$

<u>Proof :</u>

for $n \geq 1$ we have :

$$p(n) = \sum_{\overline{m} / |\overline{m}| = n} p(\overline{m})$$

$$= \sum_{k=1}^{K} q(n,k) . \sum_{\overline{m} / |\overline{m}| = n} C(\overline{m}).\overline{\alpha}^{\,\overline{m}} \frac{m_k}{n.\alpha_k}$$

and after simplification :

$$= \sum_{k=1}^{K} q(n,k) . \sum_{\overline{m} / |\overline{m}| = n-1} C(\overline{m}).\overline{\alpha}^{\,\overline{m}}$$

$$= \sum_{k=1}^{K} q(n,k) \quad \text{because} \quad \sum_{\overline{m} / |\overline{m}| = n} C(\overline{m})\overline{\alpha}^{\,\overline{m}} = \left(\sum_{k=1}^{K} \alpha_k \right)^n = 1$$

$$= q(n).$$

So, we have $p(n) = q(n)$ for $n \geq 1$, and then $p(n) = q(n)$ for all n.

<u>End of proof 2.3.</u>

As we can see, the form of the micro-states probabilities allow us to transfer computations for the multiclass model to computations for the M/H/1 single class queue. Let us show a last example of the procedure :

Corollary 2.4

Let $\Pi(\overline{z})$ be the generating function of the distribution $p(\overline{n})$, that is :

$$\Pi(\overline{z}) = \sum_{\text{all } \overline{n}} p(\overline{n}).\overline{z}^{\overline{n}} \qquad \text{where } \overline{z} \in \mathbf{C}^K \text{ and, for } k \in [1,K],\ |z_k| \leq 1$$

Then we have :

$$(2.9) \qquad \Pi(\overline{z}) = p(\overline{o}) + \sum_{k=1}^{K} \frac{z_k}{\overline{\alpha}*\overline{z}}.Q_k^*(\overline{\alpha}*\overline{z})$$

where $\overline{\alpha}*\overline{z}$ denotes the scalar product between the two vectors, that is $\sum_{k=1}^{K} \alpha_k z_k$,

and $Q_k^*(z) = \sum_{n \geq 1} q(n,k).z^n$

Proof :

$$= p(\overline{o}) + \sum_{n \neq 0} C(\overline{n}).\overline{\alpha}^{\overline{n}}\overline{z}^{\overline{n}}.\frac{1}{n}.\sum_{k=1}^{K} \frac{n_k}{\alpha_k} q(n,k)$$

$$= p(\overline{o}) + \sum_{k=1}^{K} \sum_{n \geq 1} q(n,k).z_k. \sum_{\substack{\overline{m} / |\overline{m}| = n, \\ m_k > 0}} \frac{(n-1)!\,.\,(\alpha_1 z_1)^{m_1} \ldots (\alpha_k z_k)^{m_k - 1} \ldots (\alpha_K z_K)^{m_K}}{m1! \ldots (m_k - 1)! \ldots m_K!}$$

$$= p(\overline{o}) + \sum_{k=1}^{K} z_k. \sum_{n \geq 1} q(n,k).(\overline{\alpha}*\overline{z})^{n-1}$$

$$= p(\overline{o}) + \sum_{k=1}^{K} \frac{z_k}{\overline{\alpha}*\overline{z}}. \sum_{n \geq 1} q(n,k).(\overline{\alpha}*\overline{z})^{n}.$$

End of proof 2.4.

As another application, the reader may compute, for instance, the marginal $p_k(n,n_k)$ = steady state probability that there are n customers in the system and n_k customers of class k, $n \geq n_k \geq 0$.

3. Stationary state distributions for the M/H/1 queue.

We will first explicit the form of the functions $n \mapsto q(n)$ and $(n,k) \mapsto q(n,k)$. Then we will discuss the numerical way to compute them. We will suppose here that $\mu_i \neq \mu_j$ for any phases i,j with $i \neq j$. This restriction will be easily removed later.

The balance equations for the M/H/1 model are the equations (2.4) and (2.5). If we use the probability flow conservation accross the cut $\{ (m,k) \in \mathbb{N}\times[1,K] \,/\, m \leq n \}$ (see [4], Lemma 1.4) we get the supplementary equation :

$$(3.1) \qquad \lambda.q(n) = \sum_{k=1}^{K} \mu_k.q(n+1,k) \qquad \text{for all } n$$

Using this equation in (2.6) we get :

$$(3.2) \qquad (\lambda + \mu_k).q(n,k) = \lambda.q(n-1,k) + \lambda.\alpha_k.q(n) \qquad \text{for } n \geq 1, k \in [1,K]$$

Let us introduce now the following notations :

$$\rho_k = \frac{\lambda + \mu_k}{\lambda} \qquad k \in [1,K]$$

Then, thanks to this new notations, we rewrite (2.4) and (3.2) in order to get the following system :

$$(3.3) \qquad q(o) = \sum_{k=1}^{K} (\rho_k - 1).q(1,k)$$

$$(3.4) \qquad \rho_k.q(n,k) = q(n-1,k) + \alpha_k.q(n) \qquad \text{for } n \geq 1, k \in [1,K]$$

The system $\{(3.3),(3.4)\}$ will be used to exhibit the form of the q(n)'s and the q(n,k)'s probabilities. It is easy to see that these equations are sufficient to compute them given that :

$$q(o) = 1 - \lambda.S \;;$$

remember also that (3.4) is valid for n = 1 with the convention

$$q(o,k) = \alpha_k.q(o) \qquad \text{for all k in } [1,K].$$

We need now to introduce the following elementary lemma.

Lemma 3.1

Let f be the real function defined by :

$$f : D \to \mathbf{R}$$
$$x \mapsto f(x) = \sum_{k=1}^{K} \frac{\alpha_k}{\rho_k - x}$$

with $D = \mathbf{R} - \{\rho_1, \rho_2, \ldots, \rho_K\}$,
and consider the equation :

$$f(x) = 1. \tag{3.5}$$

Then, (3.5) has exactly K different real positive solutions. If $(j_1, \ldots, j_K)$ is the permutation of $(1,2,\ldots,K)$ such that :

$$\rho_{j1} < \rho_{j2} < \ldots < \rho_{jK} \qquad (\text{ that is } \mu_{j_1} < \mu_{j_2} < \ldots < \mu_{jK})$$

and if $r_1, r_2, \ldots, r_K$ are the K solutions of (3.5) in the increasing order, then we have the following separation between the r_i's :

$$1 < r_1 < \rho_{j_1} < r_2 < \rho_{j_2} < \ldots < r_K < \rho_{jK}$$

Proof :

The proof consists of the elementary study of the function f. We have only to check the following set of properties :

* f est differentiable (infinitely differentiable) on D ,

* $\lim_{x \to +\infty} f(x) = 0^+$ and $\lim_{x \to -\infty} f(x) = 0^-$,

* $f'(x) = \sum_{k=1}^{K} \frac{\alpha_k}{(\rho_k - x)^2} > 0$ on D ,

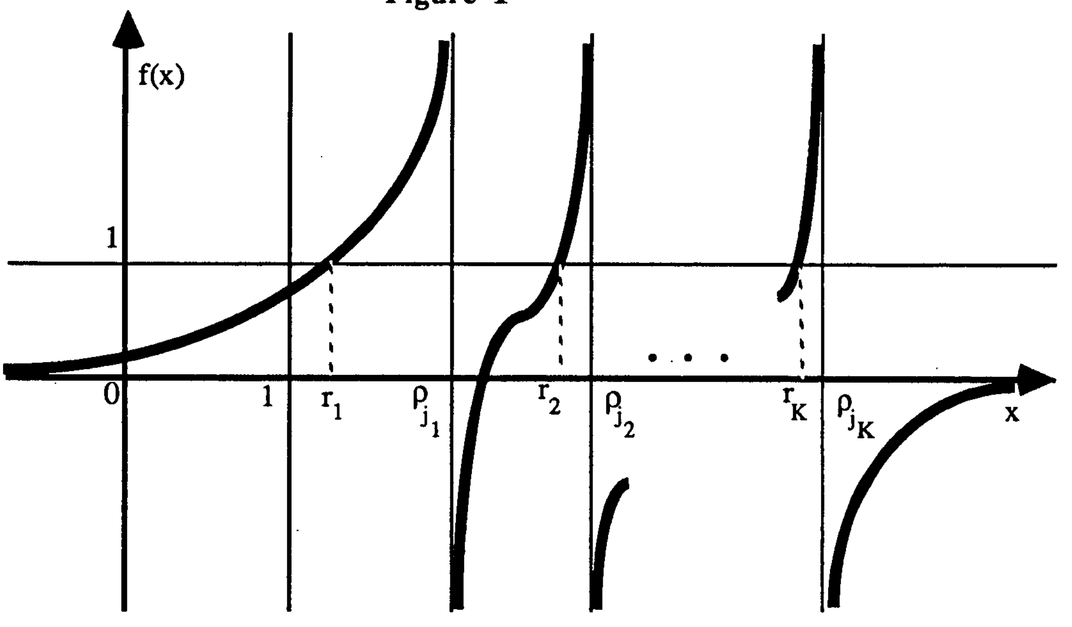
* $0 < f(0) < 1$ since $f(0) = \sum_{k=1}^{K} \frac{\alpha_k}{\rho_k} < \sum_{k=1}^{K} \alpha_k = 1$,

* $f(1) < 1$ because $f(1) = \lambda.S$,

* for $k \in [1,K]$, $\lim_{x \uparrow \rho_k} f(x) = +\infty$ and $\lim_{x \downarrow \rho_k} f(x) = -\infty$.

The set of these results allow us to determine the variation of f over D which ends the proof. The graph of the function is qualitatively represented in Figure 1.

Figure 1



End of proof 3.1.

Let Q and Q_k denote respectively the generating functions (or z-transforms) of the sequences q(.) and q(.,k) :

$$Q(z) = \sum_{n \geq 0} q(n)\, z^n \quad \text{and} \quad Q_k(z) = \sum_{n \geq 0} q(n,k)\, z^n .$$

Let us remark that the z-tranform Q_k^* introduced in Corollary 2.4 is :

$$Q_k^*(z) = Q_k(z) - q(o,k) .$$

Also, let us remark that $q(n,k) < q(n)$, so $Q_k(z)$ exists at least for $|z| \leq 1$.

By multiplying equation (3.4) by z^n and summing over n we get, after some algebra,

$$(3.6) \qquad Q_k(z) = \alpha_k . \frac{Q(z) + q(o).(\rho_k - 1)}{\rho_k - z}$$

By summing now on k and after some manipulations, we get :

$$(3.7) \qquad Q(z) = q(o).\frac{\displaystyle\sum_{k=1}^{K} \frac{\alpha_k.(\rho_k - 1)}{\rho_k - z}}{1 - f(z)}$$

We are now going to inverse Q from equation (3.7) by using Lemma 3.1. To do so, let us introduce the additional notations :

$$E(z) = \prod_{j=1}^{K} (\rho_j - z)$$

$$\text{and } E_k(z) = \prod_{\substack{j=1, \\ j \neq k}}^{K} (\rho_j - z)$$

Equation (3.7) can now be rewritten in :

$$(3.8) \qquad Q(z) = q(o).\frac{\displaystyle\sum_{k=1}^{K} \alpha_k.(\rho_k - 1).E_k(z)}{E(z) - \displaystyle\sum_{k=1}^{K} \alpha_k.E_k(z)}$$

Note that the utilization of the classical result about the M/G/1 queue :

$$Q(z) = q(o).\frac{(z - 1).H(\lambda - \lambda z)}{z - H(\lambda - \lambda z)}$$

where H(s) is the Laplace-Stieljes Transform (LST) of the hyperexponential distribution of the service time, would have given the following relation :

$$(3.9) \qquad Q(z) = q(o).\frac{(z - 1).\displaystyle\sum_{k=1}^{K} \alpha_k.(\rho_k - 1).E_k(z)}{z.E(z) - \displaystyle\sum_{k=1}^{K} \alpha_k.(\rho_k - 1).E_k(z)}$$

In (3.8) numerator and denominator of (3.9) have already been divided by $z - 1$. Equation (3.2) allows simply this simplification.

We are now ready to claim the following result :

Theorem 3.2.

$$q(n) = q(o).\sum_{k=1}^{K} c_k.r_k^{-n} \tag{3.10}$$

where $\{r_1, \ldots, r_K\}$ are the K solutions of equation (3.5),

$$\text{and } c_k = \frac{E(r_k)}{\prod_{j=1, j\neq k}^{K} (r_j - r_k)} \tag{3.11}$$

Proof :

Note first that $f(z) = \dfrac{\sum_{k=1}^{K} \alpha_k E_k(z)}{E(z)}$,

So, we have :

$$\begin{aligned}[\, z \text{ is a pole of } Q\,] &\Leftrightarrow E(z) - \sum_{k=1}^{K} \alpha_k E_k(z) = 0 \\ &\Leftrightarrow f(z) = 1 \\ &\Leftrightarrow z \in \{\, r_1, r_2, \ldots, r_K \,\}\end{aligned}$$

Remark now that $1 - f(z) = \prod_{k=1}^{K} (\rho_k - z)$ and that

$$E(r_i) = \sum_{k=1}^{K} \alpha_k.E_k(r_i) \quad \text{for } i \in [1,K] \tag{3.12}$$

Then, if N(z) is the polynomial numerator in (3.8), that is

$$N(z) = \sum_{k=1}^{K} \alpha_k.(\rho_k - 1).E_k(z)$$

we have :

$$\begin{aligned}\text{for all } i \in [1,K] \quad N(r_i) &= \sum_{k=1}^{K} \alpha_k.(\rho_k - r_i + r_i - 1).E_k(r_i) \\ &= r_i.E(r_i) \qquad \text{because of (3.12)}\end{aligned}$$

So, $N(r_i) \neq 0.$

Therefore, by partial expansion of Q, we can write :

$$Q(z) = \sum_{k=1}^{K} \frac{b_k}{r_k - z}$$

where $b_k = \lim_{z \to r_k} (r_k - z).Q(z) = q(o).\dfrac{r_k.E(r_k)}{\prod\limits_{\substack{j=1,\\ j\neq k}}^{K} (r_j - r_k)}$

Note that $b_k > 0$ for any k since for each k there are exactly k-1 negative factors both in the numerator and in the denominator of the expression above.

Last, if we write $c_k = \dfrac{b_k}{q(o).r_k}$ for $k = 1,2,\ldots,K$, we have

$$Q(z) = q(o).\sum_{k=1}^{K} \frac{c_k r_k}{r_k - z}$$

$$= q(o).\sum_{k=1}^{K} c_k \sum_{n \geq 0} r_k^{-n} z^n \qquad (|z| \leq 1)$$

and then, by identification, we get

$$q(n) = q(o).\sum_{k=1}^{K} c_k.r_k^{-n}.$$

<u>End of proof 3.2</u>.

In the same way we can explicit the probabilities q(n,k) :

<u>Theorem 3.3.</u>

(3.13) $$q(n,k) = q(o).\sum_{i=1}^{K} g_{ki}.r_i^{-n}$$

where $\{r_1, r_2, \ldots, r_K\}$ are the K solutions of equation $f(z) = 1$ and

(3.14) $$g_{ki} = \frac{\alpha_k.c_i}{\rho_k - r_i} \qquad i,k \in [1,K]$$

Proof :

The proof is quite similar to the previous one. We need only to observe that the poles of Q_k are the same as the poles of Q because of the fact that from (3.6) we can write the rational expression :

$$(3.15) \qquad Q_k(z) = q(o).\alpha_k.\frac{N(z) + (\rho_k - 1)D(z)}{(\rho_k - z)D(z)}$$

where D(z) is the denominator of (3.8). Then, it is immediat to verify that

$$N(\rho_k) + (\rho_k - 1)D(\rho_k) = 0.$$

The same technique as the one used in the previous theorem allow us to write

$$Q_k(z) = \sum_{i=1}^{K} \frac{d_{ki}}{\rho_i - z} \qquad \text{with } d_{ik} = q(o).\alpha_k.\frac{r_i c_i}{\rho_k - r_i}$$

and then the result follows by identification.

End of proof 3.3.

We can now deal with the general case for which the rates μ_k are not all different. We suppose that there exists a partition $\{ C_1, C_2, \ldots, C_M \}$ of [1,K] with $1 \leq M \leq K$ such that :

for all k in C_m, $\mu_k = \eta_m \quad 1 \leq m \leq M$, with $\eta_i \neq \eta_j$ if $i \neq j$, $i,j \in [1,M]$.

Let us note $A_m = \sum_{k / k \in C_m} \alpha_k$ for all $m \in [1,M]$.

We consider now the M/H/1 auxiliary queue with the same arrival rate λ and pdf of the service time given by

$$\sum_{m=1}^{M} A_m.\eta_m.\exp(-\eta_m t) \qquad t > 0$$

that is, an hyperexponential distribution with M phases, routing probabilities A_m and service rate η_m in phase m $(A_1 + \ldots + A_m = 1)$. We note $q^*(n,m)$ and $q^*(n)$ the associated steady state probabilities defined as before. Then, we can establish the following theorem :

Theorem 3.4.

Between the stationary distributions of the $M/H_K/1$ original queue and the $M/H_M/1$ auxiliary one (the index of the H symbol meaning for clarity the number of phases), the following relations hold :

$$(3.16) \qquad \text{(i)} \quad q^*(n) = q(n) \qquad n \in \mathbf{N},$$

$$(3.17) \qquad \text{(ii)} \quad q^*(n,m) = \sum_{k / k \in C_m} q(n,k) \qquad n \in \mathbf{N}, m \in [1,M],$$

$$(3.18) \qquad \text{(iii)} \quad q(n,k) = \frac{\alpha_k}{A_m} . q^*(n,m) \qquad n \in \mathbf{N}, k \in [1,K], m \in [1,M]$$

Proof :

Let us note $\varphi_m = \dfrac{\lambda + \eta_m}{\lambda} \qquad m \in [1,M]$

Summing the balance equations (3.4) for the original $M/H_K/1$ queue on k over C_m gives :

$$(3.19) \qquad \varphi_m . \sum_{k / k \in C_m} q(n,k) = \sum_{k / k \in C_m} q(n-1,k) + A_m . q(n) \qquad \text{for } n \geq 1$$

and for the empty state :

$$(3.20) \qquad q(o) = \sum_{k=1}^{K} (\rho_k - 1) . q(1,k)$$

which can be rewritten

$$(3.21) \qquad q(o) = \sum_{m=1}^{M} (\varphi_m - 1) . \sum_{k / k \in C_m} q(1,k)$$

We can now observe that the two M/H/1 queues are ergodic under the same condition

$$\lambda . \sum_{k=1}^{K} \frac{\alpha_k}{\mu_k} = \lambda . \sum_{m=1}^{M} \frac{A_m}{\varphi_m} < 1$$

Thus, the probabilities $q^*(n,m)$ and $q^*(n)$ defined by relations (3.16) and (3.17) are the unique solution to the fully balance equations for the auxiliary M/H/1 queue. It is easy to see that these global balance equations are equivalent to

$$\varphi_m . q^*(n,m) = q^*(n-1,m) + A_m . q^*(n) \qquad n \geq 1, \;\; m \in [1,M]$$

$$\text{and} \;\; q^*(o) = \sum_{m=1}^{M} (\varphi_m - 1) . q^*(1,m)$$

To prove (3.18) we check easily by recurrence from (3.4) that :

$$q(n,k) = \alpha_k . \left(\frac{q(o)}{\rho_k^{\,n}} + \sum_{l=1}^{n} \frac{q(l)}{\rho_k^{\,n-l+1}} \right) \tag{3.22}$$

If i and j are in C_m and if we divide q(n,i) by q(n,j), thanks to (3,22) we have :

$$\frac{q(n,i)}{q(n,j)} = \frac{\alpha_i}{\alpha_j} .$$

Together with (3.17) this relation gives :

$$q^*(n,m) = \sum_{j / j \in C_m} \frac{\alpha_j}{\alpha_k} . q(n,k) \quad \text{for any fixed k in } C_m ,$$

$$= q(n,k) . \frac{A_m}{\alpha_k} \quad \text{and so (3.18) holds.}$$

<u>End of proof 3.4.</u>

4. Stationary state distributions for the $M^K/M^K/1$ queue.

The semi-explicit form of stationary probabilities over micro-states or over regular states follow immediately from Theorem 2.1 (for micro-states) or from Corollary 2.2 (for regular states) together with the expressions obtained in Theorem 3.3 if the service rates are all different, and in Theorem 3.4 for the general case.

Another interesting distribution associated with this multiclass system is the marginal distribution of class k customers, that is, the stationnary probabilities

$p_k(n_k)$: stationary probability that there are n_k customers of the class k
for all $n_k \geq 0$.

Theorem 4.1.

If $\mu_i \neq \mu_j$ for $i \neq j$, the marginal distribution $p_k(.)$ defined above is given by :

$$(4.1) \qquad p_k(n_k) = p(o).\sum_{j=1}^{K} h_{kj}.\gamma_{kj}^{n_k} \qquad \text{for } n_k \geq 1 ,$$

$$(4.2) \qquad p_k(o) = p(o).\left(1 + \sum_{j=1}^{K} \frac{\rho_j - r_j - \alpha_k}{\rho_k - 1}.h_{kj} \right)$$

$$(4.3) \qquad \text{where } h_{kj} = \frac{\rho_k - 1}{(\rho_k - r_j).(r_j + \alpha_k - 1)}.c_j$$

$$(4.4) \qquad \text{and } \gamma_{kj} = \frac{\alpha_k}{r_j + \alpha_k - 1}$$

Proof :

The proof consists simply in tedious computations taking advantage of the linear combination of geometric terms in the expression of the distributions of the single class M/H/1 model.

The starting point is obviously the definition :

$$p_k(h) = \sum_{\overline{n} \,/\, n_k = h} p(\overline{n}) \qquad \text{for } h \in \mathbf{N} .$$

It is comfortable to use the following elementary combinatorial relations :

$$(4.5) \qquad \sum_{n \geq 0} \frac{(n+k)!}{n!}.x^n = \frac{k!}{(1-x)^{k+1}} \qquad |x| < 1 , k \in \mathbf{N}$$

and with $\overline{w} \in \mathbf{R}^K$, $w = |\overline{w}| = \sum_{k=1}^{K} w_k$, and $C(\overline{m})$ the multinomial coefficient (see Corol. 2.2) :

$$(4.6) \qquad \sum_{\substack{\overline{m} \,/\, |\overline{m}| = n, \\ m_j > 0}} C(\overline{m} - \overline{e_j}).\overline{w}^{\overline{m} - \overline{e_j}} = w^{n-1} ,$$

$$(4.7) \qquad \sum_{\substack{\overline{m} \,/\, |\overline{m}| = n, \\ m_k = p}} C(\overline{m}).\overline{w}^{\overline{m}} = \binom{n}{p}.w_k^p.(w - w_k)^{n-p} \qquad p \leq n$$

$$(4.8) \qquad \sum_{\substack{\overline{m}\,/\,|\overline{m}| = n,\\ m_j > 0\\ m_k = p}} C(\overline{m} - \overline{e_j}).\overline{w}^{\overline{m} - \overline{e_j}} = \binom{n-1}{p}.w_k^p.(w - w_k)^{n-1-p} \qquad p < n$$

See also that :

$$0 < \frac{\alpha_k}{r_j + \alpha_k - 1} < 1 \quad \text{since} \quad r_j + \alpha_k - 1 > 0 \ \ (r_j > 1 \ \text{and} \ \alpha_k - 1 > -1)$$

$$\text{and} \quad \alpha_k < r_j + \alpha_k - 1 \ \ (r_j > 1).$$

The details of the computations are omitted to discharge the text.

<u>End of proof 4.1.</u>

In the more general case where some of the service rates are eventually equal, we have, with the notations of Theorem 3.4, the following expressions :

$$q(n) = p(o).\sum_{m=1}^{M} D_m.R_m^{-n}$$

$$(4.9) \qquad p_k(n_k) = p(o).\sum_{m=1}^{M} \frac{(\varphi_h - 1).D_m}{(\varphi_h - R_m).(R_m + \alpha_k - 1)}.\left(\frac{\alpha_k}{R_m + \alpha_k - 1}\right)^n$$

$$(4.10) \qquad p_k(o) = p(o).\left[1 + \sum_{m=1}^{M} \frac{\varphi_h - R_m - \alpha_k}{(\varphi_h - R_m).(R_m + \alpha_k - 1)}.D_m\right]$$

for h such that $k \in C_h$

5. Numerical Computations.

Thanks to the results of Section 2, the only computations we have to do in order to get the stationary probabilities associated with the multiclass system are the ones necessaries to find the stationary distributions of the corresponding M/H/1 single class queue. We can first show the very simple recurrent algorithm to do this and then discuss the numerical application of the results of Section 3.

(5.1) $$q(o) := 1 - \sum_{c=1}^{K} \frac{\lambda_c}{\mu_c}$$

and for all k in [1,K] $q(o,k) := \alpha_k.q(o)$ by convention.

Let us now rewrite relation (3.4) :

(5.2) $$q(n,k) := \frac{1}{\rho_k}.q(n-1,k) + \frac{\alpha_k}{\rho_k}.q(n) \qquad n \geq 1,\ k \in [1,K]$$

then, by summation of (5.2) over k in [1,K] we get :

(5.3) $$q(n) := \sum_{k=1}^{K} \frac{q(n-1,k)}{\rho_k} + q(n).\sum_{k=1}^{K} \frac{\alpha_k}{\rho_k} \qquad n \geq 1$$

Observe that $\sum_{k=1}^{K} \frac{\alpha_k}{\rho_k} = f(0) < 1$. From (5.3) we have :

(5.4) $$q(n) = \frac{1}{1 - f(0)} \cdot \sum_{k=1}^{K} \frac{q(n-1,k)}{\rho_k}$$

which allows us to write the following recurrent algorithm :

<u>*ALGORITHM*</u> : computation of q(n) for n := 0,1,...,nmax.

Input : K ; $\lambda_1,\ldots,\lambda_K$; $\mu_1,\ldots,\mu_K$; nmax.

Output : q(n), q(n,k) for n = 0,1,...,nmax et k = 1,2,...,K.

Begin

$$q(o) := 1 - \sum_{c=1}^{K} \frac{\lambda_c}{\mu_c} ;$$

for all k *in* [1,K] *do* $q(o,k) := \alpha_k.q(o)$ *endfor* ;

for n := 1,2, ...,nmax *do*

$$q(n) := \frac{1}{1 - f(0)} \cdot \sum_{c=1}^{K} \frac{q(n-1,c)}{\rho_c} ;$$

for all k in [1,K] *do*

$$q(n,k) := \frac{q(n-1,k)}{\rho_k} + \frac{\alpha_k}{\rho_k}.q(n)$$

endfor

endfor

End

It is easy to see that if we need only the last value q(nmax), we don't have to store all the intermediate values : the only modification of the algorithm is to change q(n) and q(0) by a single scalar variable q, and to change q(o,k), q(n,k) and q(n-1,k) by an auxiliary indexed variable a[k].

Although this algorithm is straighforward, it is easily seen that for obtaining some characteristics of the multiclass system such as the marginal probabilty $p_k(0)$, it will be theorically necessary to compute an infinite sum. Instead of that, we can apply the results of Section 3, and the only numerical problem to solve then is the computation of the solutions of the equation $f(x) = 1$. To solve the equation $f(x) - 1 = 0$ we can apply the Newton-Raphson method which adapts very well to the problem and which converges very quickly.

Let us note $\rho_0 = 1$. From

$$f''(x) = \sum_{k=1}^{K} \frac{2.\alpha_k}{(\rho_k - x)^3}$$

we can see that

$$\lim_{x \downarrow \rho_{k-1}} f''(x) = -\infty \quad \text{and} \quad \lim_{x \uparrow \rho_k} f''(x) = +\infty \quad \text{for } k \in [1,K]$$

Together with the fact that f'' is increasing over $]\rho_{k-1},\rho_k[$ for $k \in [1,K]$

$$\left(\text{since } f'''(x) = \sum_{c=1}^{K} \frac{6\alpha_c}{(\rho_c - x)^4} > 0 \right)$$

this shows that f'' changes its sign only once on each interval $]\rho_{k-1},\rho_k[$; as f'' has exactly one zero in the same interval, we can first proceed to a dichotomical research to find a second interval $]a_k,b_k[$ contained in $]\rho_{k-1},\rho_k[$ for each k in [1,K] such that $f''(a_k)f''(b_k) > 0$ and $(f(a_k) - 1).(f(b_k) - 1) < 0$, and then start the Newton-Raphson iterations with for instance the middle point of $]a_k,b_k[$. If we note (x_n) the sequence which converges to a fixed solution of the equation, we have :

$$x_0 := \frac{a_k + b_k}{2} \quad \text{(if we compute the solution belonging to the interval }]\rho_{k-1},\rho_k[\text{) ;}$$

$$x_{n+1} := x_n - \frac{f(x_n) - 1}{f'(x_n)} \quad \text{for } n \geq 0$$

6. Mean values.

It is important to note that the expectations of the principal random variables associated with the $M^K/M^K/1$ system in steady state are obtained directly from the input data without any numerical intermediate computation (see for instance [5] p.76 for the response time). We are going to derive briefly here these quantities for completness. We will note (always in steady state) :

for each k in [1,K] we have :

* L_k, Q_k, U_k = resp. the mean number of class k customers in the whole system, in the waiting room and in the service center,
  ($L_k = Q_k + U_k$ and U_k is the utilization factor for class k, $U_k < 1$)

* R_k,W_k = resp. the mean response time and mean waiting time for class k customers,
  ($R_k = W_k + S_k$ where $S_k = \frac{1}{\mu_k}$)

* T_k = mean throughput in class k.

Without subscripts, all these quantities denote the same parameters without any distinction of class. For all k in [1,K] we have immediately :

$$T_k = \lambda_k \quad \text{and} \quad T = \lambda ,$$

$$L = \sum_{c=1}^{K} L_c , \quad Q = \sum_{c=1}^{K} Q_c \quad \text{and} \quad U = \sum_{c=1}^{K} U_c .$$

Then, the Little's formula applied to the whole system, to the waiting room or to the service center gives :

$$U_k = \lambda_k.S_k \quad \text{and} \quad U = \lambda S \quad (S = \sum_{c=1}^{K} \alpha_c S_c = \sum_{c=1}^{K} \frac{\alpha_c}{\mu_c}) ,$$

$$Q_k = \lambda_k W_k \quad \text{and} \quad Q = \lambda W ,$$

$$L_k = \lambda_k R_k \quad \text{and} \quad L = \lambda R .$$

From this we can also write $R = \sum_{k=1}^{K} \alpha_k R_k$

Regardless of its class, the service time of a given customer has evidently an hyperexponential distribution. A simple way to establish this formally is the following. Let us take first some notations :

C_n : class of the n^{th} arrival, $n \geq 1$,

T_n : instant of the n^{th} arrival for $n \geq 1$, $T_0 = 0$,

τ_k^n : instant of the first class-k arrival after T_{n-1}, $n \geq 1$,

$X_k^n = \tau_k^n - T_{n-1}$.

Then, a basic result about superposition of Poisson processes is : $P(C_n=k) = \alpha_k$. The proof is :

$$P(C_n = k) = P(X_1^n > X_k^n, X_2^n > X_k^n, \ldots, X_{k-1}^n > X_k^n, X_{k+1}^n > X_k^n, \ldots, X_K^n > X_k^n) =$$

$$= \int_0^{+\infty} \lambda_k \exp(-\lambda_k t). \left(\prod_{c=1}^{K} \exp(-\lambda_c t) \right) dt = \int_0^{+\infty} \lambda_k \exp(-\lambda t) \, dt = \frac{\lambda_k}{\lambda}$$

Then :

$$P(S_n < t) = \sum_{k=1}^{K} P(S_n < t / C_n = k).P(C_n = k) = \sum_{k=1}^{K} \alpha_k.\mu_k.\exp(-\mu_k t) .$$

From the results above, the waiting time distribution of class-k customers which is obviously independent of k, is the distribution of the waiting time in the single class M/H/1 associated queue. Another more formal way to establish this is the following. Since arrivals are Poisson and the K processes are independant, the so called PASTA property is satisfied (Poisson Arrivals See Time Averages, [7]) if the state of the system is a Markov process, which is true over microstates. This means for us that if we note $p^a(.)$ the steady state distribution just before an arrival, we have $p^a = p$ (for a proof valid in our context see [8]). Then, if $w_k(s)$ is the LTS of the random variable : waiting time for class-k customers, and w(s) is the corresponding LTS without any information about the class of the arriving customer, we can write :

$$w_k(s) = w(s) = \sum_{\text{all } \overline{x}} p^a(\overline{x}) f(T.\overline{x}) \qquad \text{where } f(\overline{n}) = \prod_{j=1}^{K} \left(\frac{\mu_j}{\mu_j + s} \right)^{n_j}$$

The PASTA property allows us to replace $p^a(.)$ by $p(.)$ and then, after some calculations (use the combinatorial formulae given in Theorem 4.1) we get :

$$w(s) = p(o) + \sum_{n \geq 1} \sum_{k=1}^{K} H(s)^{n-1} . \frac{\mu_k}{\mu_k + s} . q(n,k)$$

which indicates that w(s) is the LST of the waiting time in the M/H/1 associated single class queue, expressed here by conditioning with respect to the number of customers found by an arrival customer <u>and</u> the phase of the customer being served.

The mean W is given by the Pollaczec-Khinchin formula :

$$W_k = W = \frac{\sum_{c=1}^{K} U_c S_c}{1 - U}$$

See that we have then : $Q_k = \alpha_k . Q$.

Thanks to these results together with Theorem 2.1, we have that all the mean values without script are the mean values of the associated M/H/1 queue. Once the value of W is known, the previous expressions give all the expectations associated with the multiclass system.

7. Insensitivity with respect to a family of class-independant disciplines.

As Basharin showed it ([2]) as a corollary of his recurrent expressions, if we change the FIFO discipline by LIFONP (LIFO without preemption) or by RANDOM (also called SIRO, Service In Random Order, what means that the next customer to be served is choosen following an uniform distribution over the whole population in the waiting room regardless of their classes), the steady-state probabilities do not change. This result is clear from expression (2.2) since the order in the waiting room is of no importance on the micro-state stationary distribution.

We will prove this formally in a unified way and for a more general family of disciplines, since clearly the fact that allows this insensitivity property is a form of independance of the class for the queueing discipline. Associated to the $M^K/M^K/1$ system let us consider that there exists a sequence of real numbers w(m,i) belonging to the interval [0,1] with $m \geq 1$ and $1 \leq i \leq m$ such that when a customer's service ends, the next customer to be served is the i^{th} in FIFO order with probability w(m,i) if m is the number of customers in the waiting room at that moment. So, we have for all $m \geq 1$:

$$\sum_{i=1}^{m} w(m,i) = 1$$

Note that the FIFO discipline corresponds to the choice $w(m,1) = 1$ for all $m \geq 1$ (and then $w(m,i) = 0$ if $m > 1$ and $1 < i \leq m$). LIFONP corresponds to $w(m,m) = 1$ for $m \geq 1$, and the RANDOM scheduling corresponds to the choice $w(m,i) = 1/m$ for all m,i.

We can now state the result :

Theorem 7.1.

With the general discipline given above, the stationary distribution of the $M^K/M^K/1$ model over micro-states defined in Section 2 is given by (2.1) and the necessary and sufficient condition for its existence is the same as in Theorem 2.1.

Proof :

As in Theorem 2.1 and with the same arguments, we only have to verify that the function given by (2.1) satisfies the Chapman-Kolmogorov equations corresponding to the Markov process over micro-states associated to the proposed model. We will just write here the equilibrium equations :

$$p(o).\lambda = \sum_{j=1}^{K} p(j).\mu_j$$

for $n \geq 1$, $\overline{x} = (k, c_2, \ldots, c_n)$ $\quad$ ($\overline{x} = (k)$ if $n = 1$)

and $\overline{y} = (k, c_2, \ldots, c_{n-1})$ $\quad$ ($\overline{y} = (o)$ if $n = 1$)

$$p(\overline{x}).(\lambda + \mu_k) = p(\overline{y}).\lambda_{c_n} + \sum_{j=1}^{K} \sum_{i=1}^{n} p(j,c_2,\ldots,c_i,k,c_{i+1},\ldots,c_n).\mu_j.w(n,i)$$

Substituing here $p(\overline{x})$ by $\overline{\alpha}^{\,\overline{n} - \overline{e_k}}\, q(n,k)$ and

$p(j,c_2,\ldots,c_i,k,c_{i+1},\ldots,c_n)$ by $\overline{\alpha}^{\,\overline{n} + \overline{e_j}}\, q(n+1,j)$, the result follows immediately.

End proof 7.1.

8. Extension to $M^K/H^K/1$ systems.

Several of the given theorems have extensions to more general systems (see the conclusions in the next Section). To conserve the orientation of the paper to semi-explicit expressions, we just mention here the case of multiclass systems where the service distribution is hyperexponential and depends on the class of the customer. Let the pdf of the class k service time be :

$$\sum_{j=1}^{F_k} \beta_{kj}.\mu_{kj}.\exp(-\mu_{kj}t) \quad t > 0$$

where F_k is the number of phases of the class k service time distribution. Then it is easy to check that an analogous result of Theorem 2.1 holds by considering a M/H/1 single class associated queue with arrival rate λ and pdf of the service time given by :

$$\sum_{l=1}^{L} w_l.v_l.\exp(-v_l t) \quad t > 0 \quad \text{with :}$$

$$L = \sum_{j=1}^{K} F_j, \quad \text{and by noting } L_k = \sum_{j=1}^{K} F_j \ \text{ if } k \in [1,K],\ L_0 = 0,$$

then for $L_{k-1} < l \le L_k$ and $k \in [1,K]$ $w_l = \alpha_k.\beta_{k,l-L_{k-1}}$.

last, $v_l = \mu_{k,l-L_{k-1}}$.

The micro-states corresponding to the $M^K/H^K/1$ system will be of the form

$(c_1, c_2, \ldots, c_n; f)$ if $n \ge 1$, with :

c_i : class of the i^{th} customer as in Section 2,

f : phase of the customer being served, $1 \le f \le F_{c_1}$.

(plus (o) representing the empty state)

The equivalent of relation (2.1) is :

$$p(k,c_2,\ldots,c_n;f) = \overline{\alpha}^{\,\overline{n} - \overline{e}_k}\, q(n, L_{k-1} + f),$$

and all the derivations made for the $M^K/M^K/1$ system in the previous sections may be repeated here in the hyperexponential case.

9. Conclusions.

We have study a very simple system, the direct generalization of the basic M/M/1 queue to the multiclass context. The main result is the derivation of semi-explicit expressions of the most important state distributions in stationary behaviour. We show how to compute them using the obtained results and how to analytically apply these results to obtain other distributions (for instance, to multiclass queues with hyperexponential service times).

Some of the results here can be easily extended. For instance, the equivalence between the multiclass studied system and the associated M/H/1 queue holds also if there are several service centers, or if the global capacity is limited to N customers (all classes included), or to some kinds of state dependencies for arrival and service rates, or even for general service distributions (as proved for some context in [3]). In the same way, the recurrent algorithm of Section 5 is easily adapted, for instance, to Coxian distributions. The insensitivity theorem can also be extended in some of the directions presented above. We avoided to develop here these generalizations to limit the work to the M/M/1 situation, for which the study can be quite complete.

Annex : principal notations.

$\overline{x}^{\overline{y}}$: the real number $\sum_{i=1}^{N} x_i^{y_i}$, where $\overline{x}, \overline{y} \in \mathbf{R}^N$, $\overline{x} > \overline{o}$ (that is $x_i > 0$ for all i)

[1,H] : the set of integers $\{ 1, 2, \ldots, H \}$, $H \in \mathbf{N}$

$\overline{e_i}$: the i^{th} vector of the canonical base of $\mathbf{R}^N$ for some $N \in \mathbf{N}$,
that is the vector $(0,0,\ldots,0,1,0,\ldots,0)$ where the 1 arrives in position i

$C(\overline{m})$: the multinomial coeficient $\dfrac{m!}{m_1!.m_2!. \ldots .m_H!}$ for some $H \in \mathbf{N}$

K : number of classes in a multiclass system

[1,K] : set of classes $\{1,\ldots,K\}$

λ_k : arrival rate of class k customers

$\mu_k = \dfrac{1}{S_k}$: service rate of class k customers

$\lambda = \sum_{k=1}^{K} \lambda_k$: total arrival rate

α_k : the ratio $\dfrac{\lambda_k}{\lambda}$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K)$